An Analytical Investigation of the Bullwhip Effect
NTC S03-MD13s

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GOALS

Companies attempt to minimize inventory while maintaining sufficient to guard against fluctuations in demand. We have found new, exact analytical solutions to the full set of supply chain equations, opening up a whole new range of research opportunities. We intend to use these new solutions to develop optimal ordering policies that reduce the Bullwhip Effect, and generally improve inventory management. The goals of the research are:

1) To advance the mathematical understanding of the new solutions, both theoretically and through numerical integrations.
2) To broaden the set of solutions to cover a wide range of textile and apparel inventory problems.
3) To produce algorithms, heuristics, and guidelines for improving inventory management software systems.

ABSTRACT

The bullwhip effect is a well-established, frequent, and expensive occurrence: modest fluctuations in consumer demand are dramatically amplified as they proceed up the supply chain from retailer to apparel manufacturer to textile manufacturer. The Bullwhip Effect is a significant problem in wide variety of companies and industries. We have used our new solutions to the supply chain equations to develop optimal ordering policies that reduce the Bullwhip Effect, and generally improve inventory management. A surge in demand depletes inventory, and we have determined the critical parameters that replenish the inventory without any overshoot. This is a valuable tool, because retailers and manufacturers can adjust their orders to return their inventories to the desired level. Managerially useful ordering strategies have emerged from the mathematics. For example, the theoretical solutions allow us to eliminate long-term inventory deficits.

We have exactly solved the supply chain equations, and calibrated the Bullwhip Effect.

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2004 Annual Report

This annual report covers the period May 1,2003 through May 1, 2004. This is a seed project with duration one year. The following papers arising from this project provide additional reading:

Warburton Roger D. H., 2004. **An analytical investigation of the Bullwhip Effect**, International Journal of Production and Operations Management, in press.
Warburton Roger D. H., 2004. **An exact solution to the production inventory control problem**, International Journal of Production Economics, 92, 1, 81-96.
Hodgson, J. P. E. and R. D. H. Warburton, 2004. **On solutions to the linear delay equation in supply chain modeling**, submitted to SIAM (Society for Industrial and Applied Mathematics) Journal on Applied Mathematics.

1. Introduction: The Bullwhip Effect

Lee et al. popularized the term "Bullwhip Effect" where a retailer's orders to their suppliers tend to have a larger variance than the consumer demand that triggered the orders [13][14]. This demand distortion propagates upstream with amplification occurring at each echelon. Lee et al. identified four major causes of the Bullwhip Effect: users interpreting orders (the demand); order batching; promotions, which artificially stimulate demand; and supply shortages, which also lead to artificial demands. Sterman demonstrated that the Bullwhip Effect is a significant problem in an experimental, managerial context [16], and has been documented in a wide variety of companies and industries [15][12].

Much earlier, however, Forrester had defined a simplified form for the equations describing the relation between inventory and orders and pioneered the simulation approach [10]. Burbidge emphasized the now well-accepted principles of cycle time reduction and order synchronization [1], and later coined his Law of Industrial Dynamics: "If demand is transmitted along a series of inventories using stock control ordering, then the demand variation will increase with each transfer" [2].

1.1 Related theoretical analyses

Kahn showed that a serially correlated demand results in the Bullwhip Effect [11]. Lee et al. employed the same demand assumption, and used a cost minimization approach to show that distortion in demand arises when retailers optimize orders, and that amplification increases as the replenishment lead-time increases [13][14]. Various demand distributions and numerical experiments have been employed to study the Bullwhip Effect, e.g., Chen analyzed the impact of exponential smoothing [3].

Disney et al. provide a useful compilation of the control theory literature applicable to the Bullwhip Effect [5]. Particularly relevant is that John et al. proved that for step function shocks to the inventory, a long-term inventory deficit could occur [9].

We treat time as a continuous variable, which is appropriate if production and distribution orders are updated whenever new demand information becomes available. The evolution towards Just in Time (JIT) manufacturing with its requirement for continuous replenishment increases the significance of continuous time models. We have published two papers on our analysis of the Bullwhip effect [18][19].

2. The Retailer's Supply Chain

Retailers attempt to minimize their inventory while maintaining sufficient on hand to guard against fluctuations in demand. The inventory, *I(t)*, is depleted by the demand rate, *D(t)*, and increased by the receiving rate, *R(t)*, so the inventory balance equation is:

$$\frac{dI}{dt} = R(t) - D(t) \tag{1}$$

For the demand term, we analyzed a step function surge in demand, which is a rich source of insight when seeking an understanding of the trade-offs involved in tuning an ordering policy. Also, since the equations are linear, any arbitrary demand can be built from a suitable linear combination of step functions [18]. The lead-time, or production delay, τ, is the time from the issue of orders until the receipt of the goods from the supplier. Thus, the receipts are equal to the orders placed at a previous time, and $R(t) = O(t - \tau)$.

When a retailer orders from a manufacturer who employs a “make-to-stock” policy, the order may be fulfilled from the manufacturer’s inventory, in which case the fulfillment delay appears to the retailer as just the sum of the order processing and shipping times. If the manufacturer employs a “make-to-order” policy, then the fulfillment time will be much longer. In neither case can the retailer easily change the value of the fulfillment delay, τ, which we can therefore consider to be a constant. While the retailer is often considered to be driving the supply chain, it is the manufacturer who determines the fulfillment time. The retailer or manufacturer can affect the lead time through long term, strategic moves, such as changing from offshore to onshore, quick response manufacturing, or to lean manufacturing, which shortens the manufacturing cycle [20]. However, the fulfillment delay is not tunable in the sense that one can rapidly vary it to achieve a particular ordering goal.

2.1 The Ordering Policy

The goal of an ordering policy is to bring the actual inventory towards the desired inventory. The policy we analyzed is a generalized Order-Up-To (OUT) policy, defined as:

$$O(t) = (I_o - \text{ inventory position}) + \text{ perceived demand} \tag{2}$$

O(t) is the ordering decision made at time, *t*, in which I_o is the desired order-up-to level. The inventory position equals net stock plus inventory on order, which can also include items in manufacturing -- the work-in-process (WIP). One must also order an amount corresponding to the current, perceived demand. If the ordering policy reacts too quickly to changes in the actual, instantaneous demand, instability results. Therefore, in practice the actual demand is smoothed out into the perceived demand.

Sterman used a simplified beer production and distribution system to demonstrate that managers only poorly understand supply chain concepts, and identified a set of heuristics that humans use to place orders, based on 2,000 sets of results [16]. Sterman's heuristics can be directly related to the three parameters used here, and therefore, the above, three-parameter model covers a wide variety of realistic supply chain problems [9].

2.2 Demand contribution

It is usually suggested that some kind of smoothing should be applied to the demand data. Otherwise, excessive fluctuations occur resulting in increased production costs. Exponential smoothing is easy to implement and relatively accurate for short-term forecasts. The tunable parameter, T_a, controls the amount of smoothing to be applied to the raw demand. The smoothed demand contribution to the order rate is:

$$O_d(t) = D_o + d\{1 - exp(-t / Ta)\} \quad (3)$$

2.3 Inventory replenishment contribution

The goal of the inventory replenishment term in the ordering policy is to bring the actual inventory towards the desired inventory:

$$O_i(t) = \frac{I_o - I(t)}{T_i} \quad (4)$$

I_o represents the desired inventory. This policy has the advantage that it replaces deficits due to a surge in demand, and the tunable parameter, T_i, acknowledges that the deficit recovery should be spread out over time [17].

2.4 WIP contribution

The ordering policy also depends on the WIP on the shop floor. If a surge occurs, the WIP will be depleted, and it is desirable to increase the order rate to stop the decline. On the other hand, excessive WIP decreases the ordering rate [7]. The WIP contribution, O_w, is:

$$O_w(t) = (1 / T_w)\{WIP_o - WIP(t)\} \quad (5)$$

The WIP is the sum of the unfulfilled (i.e., undelivered) orders:

$$WIP(t) = \int_{t-\tau}^{t} O(t)dt \quad (6)$$

The tunable parameter, T_w, allows the order rate to depend on the quantity of WIP on the shop floor. WIP_o represents the desired value of the WIP, which is a function of the fulfillment delay and the demand, i.e., there is enough work on the shop floor to satisfy the demand for the duration of the fulfillment time. Therefore, the term in the order rate due to the WIP deficit is:

$$O_w(t) = (1/T_w)\left\{\tau D_o - \int_{t-\tau}^{t} O(t)dt\right\} \tag{7}$$

The parameters T_a, T_i, and T_w provide great flexibility in shaping the inventory's dynamic response: T_a controls the smoothing of the demand; T_i adjusts the rate at which the inventory deficit is recovered; and T_w adjusts the WIP replenishment rate.

Using these order rates in the inventory balance equation gives the system to be solved. The method of the exact solution is outlined in section 4. The flavor of the solutions can be seen in Figure 1, where it is clear that the response of the inventory depends sensitively to the ratio of the replenishment delay, τ, to the inventory deficit parameter, T_i. Larger replenishment delays increase the divergence of the inventory response (i.e., dramatic overshoots). However, the parameter, T_i, can be tuned so that the inventory returns exactly to the desired level without overshoot. The inventory response for the critical value, T_i^* is shown in the figure.

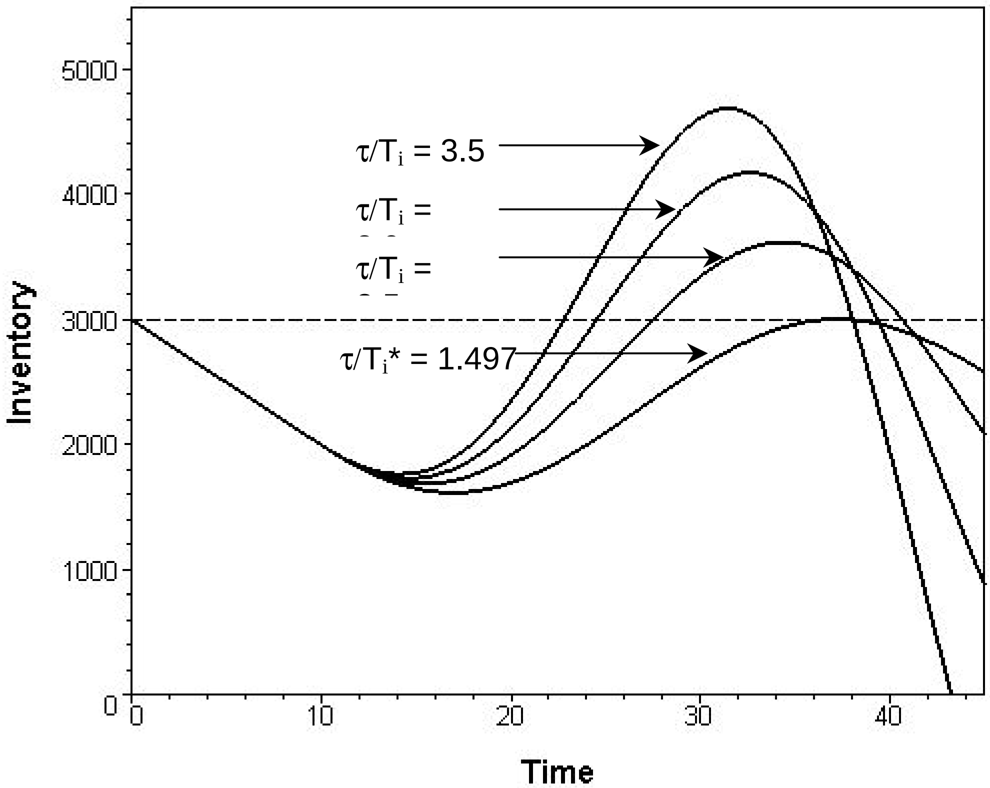

Figure 1: Four exact solutions of the inventory equation. The surge in demand depletes the inventory for the duration of the replenishment delay ($\tau = 10$). However, the critical adjustment rate, T_i^*, brings the inventory exactly back to its desired value.

3. Analysis of the Bullwhip Effect

The Bullwhip Effect is defined as the amplification of order variability along the supply chain. Since we have the exact solution for the inventory (see Figure 1), we also have the exact solution for the order rate. Figure 2 shows an example of a plot of the retailer's order rate responding to a surge in demand. The retailer's order rate quickly grows to exceed the constant consumer demand rate. However, the orders reach a peak, and quickly decline. Meanwhile, the retail orders deplete the manufacturer's inventory, and the manufacturer issues orders to the supplier. These orders grow even more dramatically, as seen in Figure 2.

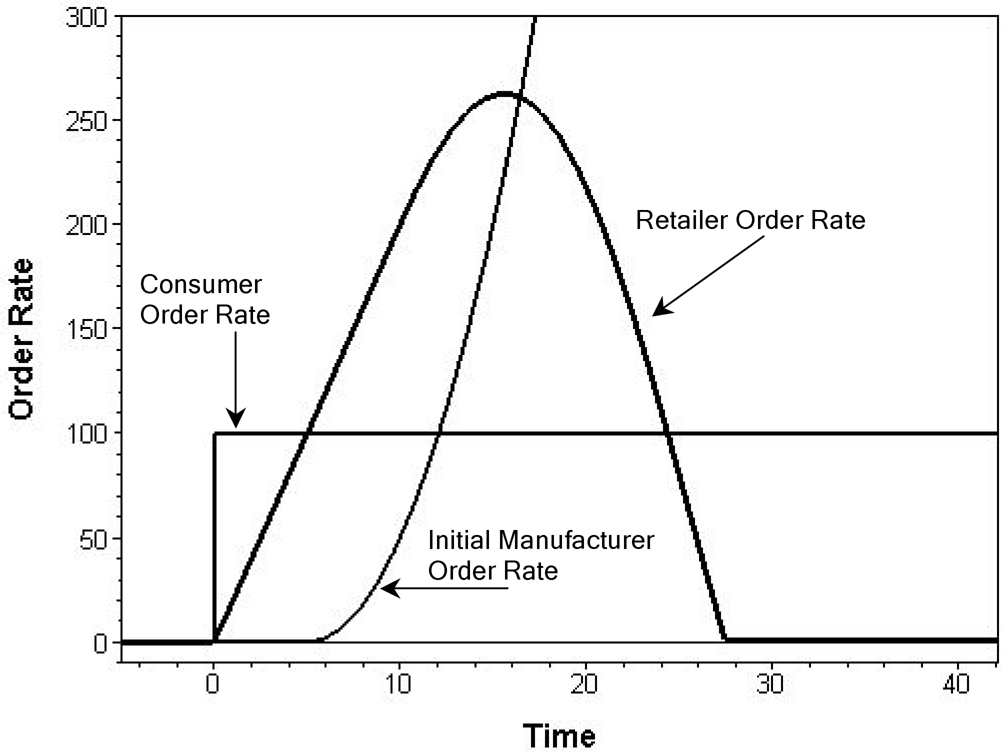

Figure 2. The "Bullwhip Effect," the amplification of order rate from consumer to retailer to manufacturer. The flood and drought in retail orders is also apparent.

3.1 Measuring the Bullwhip Effect

One measure of the Bullwhip Effect is the ratio of the output order rate (retailer orders to manufacturer) to the input order rate (consumer demand). The retail order rate climbs to a peak, which occurs soon after τ, and so this is an appropriate time at which to compare the rates:

$$BW^R = \frac{O(\tau)}{d} = \frac{I_0 - I(\tau)}{T_i d} = \frac{\tau}{T_i} \tag{8}$$

The superscript, *R*, indicates that this amplification in orders is attributable to the retailer's ordering policy. Since we have analytical solutions, the relative contributions of the parameters are explicit. For example, equation (8) is independent of the size of the surge, which is a direct consequence of defining the Bullwhip Effect as a ratio. Equation (8) also suggests that increasing the value of T_i can reduce the Bullwhip Effect. However, equation (16) tells us that T_i cannot be raised arbitrarily without permanent inventory deficits occurring. If it is important for the inventory to return to its desired value, then some Bullwhip Effect is inevitable, e.g., T_i^* is the logical choice because it returns the inventory to its desired value without overshoot, and for $\tau = 10$, $T_i^* = 6.68$, and $BW^R = 1.5$.

In the "best" replenishment scenario, the Bullwhip is 50%!

The definition in equation (8) only represents the orders in the early stages of the inventory cycle, and does not account for the inventory overshoot. A different calibration of the Bullwhip Effect measures the growth in inventory fluctuations, rather than order fluctuations. For example, in Figure 2 one can compare the peak in the inventory (overshoot) to the bottom of the decline (undershoot). This measure of the Bullwhip Effect has a different character from that in equation (8), because the behavior of the inventory is different from that of the orders. For example, in the critical T_i^* case, the overshoot was adjusted to be zero, and so $BW_I^R = 0$.

In other words, if the Bullwhip in inventory is adjusted to zero, the Bullwhip in orders may remain. If it is more important to replenish the inventory than to minimize order fluctuations, then equation (9) is a better measure of the impact of demand fluctuations. Bullwhip analyses typically concentrate on order variability, and while there is clearly a cost benefit to its reduction, other impacts, such as inventory replenishment, should be considered. For example, customer service levels will depend on the ability to replenish the inventory [8].

3.2 The Manufacturer's Bullwhip

We have also studied the impact on the manufacturer of the orders from the retailer. The manufacturer's situation is frequently more complicated than the retailer's because orders to suppliers often represent sub-component orders rather than complete items. Typically, the sub-components have different manufacturing and shipment delays. Additionally, manufacturers usually supply several retailers. However, the general approach used in the retail case still applies: The manufacturer creates an ordering policy for each item, and the receipts from the supplier will be characterized by the supplier's replenishment delay time for that item.

The exact solution to the manufacturer's equations [18] shows that the impact of the retailer's orders is to cause the manufacturer's inventory to decline as a quadratic function of time. This is in contrast to the linear decline in the retailer's inventory -- see Figure 2. Also, the manufacturer's order rate to the supplier initially grows as a quadratic function of time. This is in stark contrast to the retailer's order rate, which initially grows linearly, and the constant consumer demand rate, which started the whole order train -- see Figure 2. The Bullwhip Effect clearly emerges. Further, the manufacturer does not realize that he is witnessing the flood part of the order cycle from the retailer, and that a drought is to follow.

3. The Exact Solution to the Supply Chain Equations

The order rate and inventory equations make up the system to be solved:

$$\frac{dI}{dt} - O(t-\tau) = -(D_o + d) \tag{9}$$

$$O(t) + \frac{1}{T_w}\int_{t-\tau}^{t} O(t)dt + \frac{I(t)}{T_i} = \frac{I_o}{T_i} + \frac{\tau D_o}{T_w} + D_o + d - d\,exp(-t/T_a) \tag{10}$$

Equations (9) and (10) can be solved exactly in terms of the Lambert W function. Corless et al. provide a review of the history, theory, and applications of the Lambert W function [4]. The Lambert W function is multi-valued with an infinite number of branches, but fortunately, it is readily available in efficient and accurate implementations, such as in Maple, where it is defined as LambertW, and Mathematica, where it is referred to as the Product Log function. The result is that the entire, exact solution for the inventory and orders for $t \geq \tau$, is:

$$I(t) = A' exp\left[qt\right] + I_o - \frac{d\tau T_i}{T_w} + Rd\,exp\left[-(t-\tau)/T_a\right] \tag{11}$$

$$O(t) = qA'e^{q(t+\tau)} + D_o + d - (dR/T_a)exp(-t/T_a), \tag{12}$$

where

$$q = \frac{W(z)}{\tau} - \frac{1}{T_w} \qquad z = -\frac{\tau}{T_i}exp(\tau/T_w)\left(1 - \frac{T_i}{T_w}\right) \qquad A' = a + i\alpha \qquad W(z) = \omega + i\Omega$$

$$R^{-1} = \left\{\left(\frac{1}{T_a} - \frac{1}{T_w}\right) + \left(\frac{1}{T_w} - \frac{1}{T_i}\right)exp(\tau/T_a)\right\} \qquad a = \frac{VY - U\sin\Omega}{\Omega} \qquad \alpha = \frac{VX - U\sin\Omega}{\Omega}$$

$$X = (\omega - \tau/T_w)\cos\Omega - \Omega\sin\Omega \qquad Y = (\omega - \tau/T_w)\sin\Omega + \Omega\cos\Omega$$

$$U = -d\tau e^{-\omega} e^{\tau/T_w}(1 - R/T_a) \qquad V = d\,e^{-\omega} e^{\tau/T_w}(\tau T_i/T_w - R - \tau)$$

3.1 Properties of the solution: Inventory Deficits

The analytical solutions allow us to investigate the properties and characteristics of the above solution. For example, we can eliminate the exponential smoothing and WIP terms by employing the limits, $T_a,\ T_w \to \infty$. Only the inventory deficit ordering policy term remains. For small, negative values of τ/T_i, the term in $exp(Wt/\tau)$ decays to zero, and the inventory approaches a constant -- the stable regime, where $I(t) \to I_o - dT_i$. That is, ***permanent*** inventory deficits occur in the stable regime. This confirms a similar result previously proved by John et al., using the Final Value Theorem [9].

3.2 Managing the Inventory

In most industrial applications, the fulfillment time, τ, is fixed (at least on average) and is not easily tunable. However, the parameter T_i is adjustable, and can be tuned to eliminate the inventory overshoot – see Figure 1. Since the solutions are analytical, it is straightforward to compute for any τ, the value of T_i that brings the inventory back precisely (and only) to the desired level, I_o. As shown in Figure 1, for any delay time, τ, the value of $T_i = T_i^*$ returns the inventory to its desired level exponentially fast without any overshoot.

When a retailer detects a surge in consumer demand, the time of the peak in inventory can be calculated. If the size of the surge in demand is estimated, the retailer can adjust the orders so that the inventory is made up without suffering either deficits or overshoots. This provides an example of the power of the analytical solutions. Surges can significantly deplete the inventory, increasing the probability of stockouts and backlogs. The parameter T_i can be adjusted to balance the deficit recovery against the backlog.

4. Conclusions

We have demonstrated that it is possible to solve exactly the supply chain equations. No approximations were required. The replenishment delay emerges as responsible for much of the rich and complex behavior associated with the inventory response. We have also verified the solutions by comparison with direct numerical integration and established that the theoretical solutions provide an excellent representation of the inventory behavior.

The ordering policy was parameterized so as to allow the retailer to calculate a critical order adjustment rate that returns the inventory back to its desired value exponentially fast, while not generating an overshoot.

We calculated the replenishment rate for orders issued by the retailer to the manufacturer, and solved the equation for the manufacturer's inventory reacting to the retailer's orders. The Bullwhip Effect emerged naturally, and its size was calculated. The additional Bullwhip Effect attributable to the manufacturer was also calculated. It emerged that one must trade the reduction in the Bullwhip Effect against competing processes such as permanent inventory deficits, or inventory excesses.

Web Site:
http://www.warbs.net/bullwhip.

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